

BOUNDARY OF THE RELATIVE OUTER SPACE

Erika Meucci

December 2, 2011

Abstract

Let $\mathcal{A} = \{A_1, \dots, A_k\}$ be a system of free factors of F_n . The group of relative automorphisms $\text{Aut}(F_n; \mathcal{A})$ is the group given by the automorphisms of F_n that restricted to each A_i are conjugations by elements in F_n . The group of relative outer automorphisms is defined as $\text{Out}(F_n; \mathcal{A}) = \text{Aut}(F_n; \mathcal{A})/\text{Inn}(F_n)$, where $\text{Inn}(F_n)$ is the normal subgroup of $\text{Aut}(F_n)$ given by all the inner automorphisms. This group acts on the relative outer space $\text{CV}_n(\mathcal{A})$. We prove that the dimension of the boundary of the relative outer space is $\dim(\text{CV}_n(\mathcal{A})) - 1$.

1 Introduction

Let F_n denote the free group of rank n . We consider the group of automorphisms of F_n , denoted by $\text{Aut}(F_n)$, and the group of outer automorphisms

$$\text{Out}(F_n) = \text{Aut}(F_n)/\text{Inn}(F_n),$$

where $\text{Inn}(F_n)$ is the normal subgroup of $\text{Aut}(F_n)$ given by all the inner automorphisms. In 1986 Culler and Vogtmann introduced a space CV_n on which the group $\text{Out}(F_n)$ acts with finite stabilizers and proved that CV_n is contractible. That space CV_n is called *outer space*. In [5], Gaboriau and Levitt computed the dimension of the boundary of outer space. We are interested in studying particular subgroups of $\text{Out}(F_n)$.

Let $\mathcal{A} = \{A_1, \dots, A_k\}$ be a system of free factors of F_n , that is, there exists $B < F_n$ such that $F_n = A_1 * \dots * A_k * B$. We define the group of relative (to \mathcal{A}) automorphisms $\text{Aut}(F_n; \mathcal{A})$ given by the elements $f \in \text{Aut}(F_n)$ such that f restricted to each A_i is a conjugation by an element in F_n . Note that $\text{Aut}(F_n) > \text{Aut}(F_n; \mathcal{A}) \triangleright \text{Inn}(F_n)$. We define also the group of relative outer automorphisms:

$$\text{Out}(F_n; \mathcal{A}) = \text{Aut}(F_n; \mathcal{A})/\text{Inn}(F_n) < \text{Out}(F_n).$$

In [10], it was introduced a contractible space, that we will denote by $\text{CV}_n(\mathcal{A})$, on which $\text{Out}(F_n; \mathcal{A})$ acts. This space is called relative outer space and it can be thought as a subset of the compactification of CV_n . In [10], we proved that an irreducible relative outer automorphism with irreducible powers acts on the compactification of the modified relative outer space with north-south dynamics. A natural question arose: what is the dimension of the boundary? Our goal is to give an answer to that question.

Main Theorem. The dimension of $\partial\text{CV}_n(\mathcal{A})$ is equal to $\dim(\text{CV}_n(\mathcal{A})) - 1$.

The main ingredient in the proof of the Main Theorem is the computation of the \mathbb{Q} -rank of a tree (with special points). In Section 2, we start defining the relative outer space and then we review some of its properties studied in [10]. In Section 3, we give the definition of geometric and non-geometric trees with special points and we discuss some features of the geometric trees that are a relative version of the facts in [5]. The goal of Section 4 is to define the index of a tree with special points. An upper bound of the index will be a key step in the computation of the \mathbb{Q} -rank of a tree with special points and hence in the proof of the Main Theorem in Section 5.

Acknowledgments

I wish to thank Mladen Bestvina for his suggestions and very useful conversations.

2 Relative Outer Space

In this section we define the relative outer space and its boundary. Moreover, we review some of the properties of the relative outer space studied in [10].

Let $\mathcal{A} = \{A_1, \dots, A_k\}$ be a system of free factors of F_n , that is, there exists $B < F_n$ such that $F_n = A_1 * \dots * A_k * B$.

Consider $A_i = \langle y_1^i, \dots, y_{s(i)}^i \rangle$ and $F_n = \langle y_1^1, \dots, y_{s(k)}^k, x_1, \dots, x_{n-\sum_{i=1}^k s(i)} \rangle$. By a graph we mean a connected 1-dimensional CW complex. Let the *relative rose* $R_n(\mathcal{A})$ be a graph obtained by a wedge of $n - \sum_{i=1}^k s(i)$ circles $e_1, \dots, e_{n-\sum_{i=1}^k s(i)}$ attaching $\sum_{i=1}^k s(i)$ circles $C_1^1, \dots, C_{s(k)}^k$ on k stems, where the i th stem has $C_1^i, \dots, C_{s(i)}^i$ attached to it. Moreover,

$$\pi_1(R_n(\mathcal{A}), v) \cong F_n = \langle y_1^1, \dots, y_{s(k)}^k, x_1, \dots, x_{n-\sum_{i=1}^k s(i)} \rangle,$$

where v is the vertex in $R_n(\mathcal{A})$ intersection of the circles $e_1, \dots, e_{n-\sum_{i=1}^k s(i)}$, by declaring y_i^j to be the homotopy class of C_i^j and x_i to be the homotopy class of the loop e_i . Let $(R_n(\mathcal{A}), \underline{k})$ be the graph $R_n(\mathcal{A})$ equipped with inclusions $k_j : \bigvee_{i=1}^{s(j)} S^1 \rightarrow R_n(\mathcal{A})$ that identifies $\bigvee_{i=1}^{s(j)} S^1$ with $\bigvee_{i=1}^{s(j)} C_i^j$, for all $j = 1, \dots, k$.

Definition 2.1. Let Γ be a graph of rank n with vertices of valence at least 3, equipped with embeddings $l_j : \bigvee_{i=1}^{s(j)} S^1 \rightarrow \Gamma$ for $j = 1, \dots, k$. We call $\mathbb{B}_j = l_j(\bigvee_{i=1}^{s(j)} S^1)$ *wedge cycle*. The *dual graph* of the \mathbb{B}_j 's is the graph with one vertex for each wedge cycle, one vertex w for each intersection between two or more wedge cycles and edges between w and vertices corresponding to the wedge cycles meeting in w .

Definition 2.2. An (\mathcal{A}, n) -graph (Γ, \underline{l}) is a finite graph Γ of rank n with vertices of valence at least 3, with possible separating edges, equipped with embeddings $l_j : \bigvee_{i=1}^{s(j)} S^1 \rightarrow \Gamma$ for $j = 1, \dots, k$, such that any two \mathbb{B}_j intersect in at most a point and the dual graph of the \mathbb{B}_j 's is a forest.

Notation 2.3. We denote by \mathcal{A} the set of free factors A_1, \dots, A_k and we will denote $A_1 * \dots * A_k * B$ by $\mathcal{A} * B$. Sometimes we will also write \mathcal{A} for the subgroup $A_1 * \dots * A_k$. The meaning of \mathcal{A} should be clear from the context.

Definition 2.4. A *marked metric* (\mathcal{A}, n) -graph (Γ, ϕ, l) is a marked graph (with possible separating edges) (Γ, ϕ) such that

- each edge e in $\widehat{\Gamma}$ (the graph obtained from Γ by collapsing the wedge cycles to special points) has length $\widehat{l}(e) = l_{\widehat{\Gamma}}(e) \in (0, 1]$ and each edge in a wedge cycle has length 0;
- the wedge cycles are disjoint.

Definition 2.5. The *relative outer space* $\text{CV}_n(\mathcal{A})$ is the space of equivalence classes of marked metric (\mathcal{A}, n) -graphs where

1. the sum of all lengths of the edges in $\widehat{\Gamma}$ is 1 (relative volume 1);
2. $(\Gamma_1, \phi_1, l_1) \sim (\Gamma_2, \phi_2, l_2)$ if there is an isometry $h : \Gamma_1 \rightarrow \Gamma_2$ such that $h \circ \phi_1(C_i^j) = \phi_2(C_i^j)$, $\forall i, j$ and $h \circ \phi_1$ is homotopic to ϕ_2 rel. C_i^j , $\forall i, j$.

Remark 2.6. This space (modulo possible edges of length 0 not in the wedge cycles) was introduced in the third chapter of [10], and it was called modified relative outer space to distinguish this space from another space, called relative outer space, introduced in the second chapter.

Let $R_n(\mathcal{A})$ be the relative rose with each edge in the wedge cycles of length 0. There is a natural right action of $\text{Out}(F_n; \mathcal{A})$ on $\text{CV}_n(\mathcal{A})$ given by changing the marking: let $X = (\Gamma, \phi, l) \in \text{CV}_n(\mathcal{A})$ and $\Psi \in \text{Out}(F_n; \mathcal{A})$, consider $\psi : R_n(\mathcal{A}) \rightarrow R_n(\mathcal{A})$ such that it fixes the wedge cycles and $[\psi_*] = \Psi$. The right action is given by

$$X \cdot \Psi = (\Gamma, \phi, l) \cdot \Psi = (\Gamma, \phi \circ \psi, l).$$

Notice that the stabilizer of a point may be infinite. We define a topology on $\text{CV}_n(\mathcal{A})$ by varying the length of the edges that are not in any wedge cycle. Since the sum of the lengths of these edges is 1, $\text{CV}_n(\mathcal{A})$ is a simplicial complex with missing faces. We define the *relative spine* $S_n(\mathcal{A})$ of the relative outer space as the geometric realization of the partially ordered set of open simplices. Notice that $S_n(\mathcal{A})$ is a simplicial complex.

In [10], we proved the following result.

Theorem 2.7. The relative outer space $\text{CV}_n(\mathcal{A})$ is contractible.

In [10], we computed the dimension of $\text{CV}_n(\mathcal{A})$ and the dimension of $S_n(\mathcal{A})$. We briefly sketch the proof of such computation. Suppose that Γ is a maximal graph in $S_n(\mathcal{A})$, i.e., it has the maximum number of vertices, and consider $\widehat{\Gamma}$. Denote by V and E the number of vertices and edges of $\widehat{\Gamma}$ respectively. The vertices corresponding to the special points have valence 1 and the remaining vertices have valence 3. Hence,

$$E = \frac{3(V - k)}{2} + \frac{k}{2}.$$

Because $V - E = 1 - (n - \sum_{i=1}^k s(i))$, we have

$$V = 2n + 2k - 2 - 2 \sum_{i=1}^k s(i)$$

and hence, because we can collapse V vertices to $s = \max\{k, 1\}$ vertices and the wedge cycles are disjoint,

$$\dim(S_n(\mathcal{A})) = 2n + 2k - 2 - 2 \sum_{i=1}^k s(i) - s.$$

Moreover,

$$E = 3n + 2k - 3 - 3 \sum_{i=1}^k s(i)$$

and because the relative volume of each graph is 1,

$$\dim(\text{CV}_n(\mathcal{A})) = 3n + 2k - 4 - 3 \sum_{i=1}^k s(i).$$

Notice that if $k = 0$, $\dim(\text{CV}_n(1)) = 3n - 4 = \dim(\text{CV}_n)$. Indeed, if $k = 0$ the relative outer space is the standard outer space.

Example 2.8. Consider $\text{Out}(F_2; A)$, where $F_2 = \langle a, b \rangle$, $A = \langle a \rangle$. In that case, $\text{Out}(F_2; A)$ is isomorphic to the infinite dihedral group D_∞ (see [10]). The relative outer space $\text{CV}_2(A)$ is a point X with an infinite countable number of half-open edges attached to it. The action of the group on $\text{CV}_2(A)$ is given by rotating the edges. Hence, the stabilizer of X is $\text{Out}(F_2; A)$. Moreover, $\text{CV}_2(A)/\text{Out}(F_2; A)$ is a 1-simplex with a missing vertex. The relative spine $S_2(A)$ is a point with an infinite number of closed 1-simplices coming out from that point. The relative outer space $\text{CV}_2(A)$ is not locally compact and this is true in general: $\text{CV}_n(\mathcal{A})$ is not locally compact if $\mathcal{A} \neq 1$.

We can think of the relative outer space as a deformation space of trees. We recall the main definitions and some results for actions on \mathbb{R} -trees.

Definition 2.9. Let (X, d) be a metric space. We say that (X, d) is an \mathbb{R} -tree if for any $x, y \in X$ there is a unique arc from x to y and this arc is a geodesic segment.

Let $\phi : T \rightarrow T$ be an isometry of an \mathbb{R} -tree T . The *translation length* of ϕ is

$$l(\phi) = \inf\{d(x, \phi(x)) \mid x \in T\}.$$

The infimum is always attained and there are two possible cases. If $l(\phi) > 0$, there is a unique ϕ -invariant line called the axis of ϕ , and $\phi|_{\text{axis}}$ is a translation by $l(\phi)$. In this case, we say that ϕ is *hyperbolic*. If $l(\phi) = 0$, then ϕ fixes a non-empty subtree of T and is said to be *elliptic*.

Let G be a group acting by isometries on an \mathbb{R} -tree T . A tree equipped with an isometric action is called *G-tree*. The action is *nontrivial* if no point of T are fixed by the whole group. It is *minimal* if there is no proper G -invariant subtree. The action is *free* if any nonidentity group element does not leave an element of T fixed. Let $Gx = \{gx \mid g \in G\}$ be the orbit of $x \in T$. An action of G on T has *dense orbits* if the closure of Gx is the whole tree T . A map $f : T \rightarrow T'$ between \mathbb{R} -trees is a *morphism* if every segment in T can be written as a finite union of subsegments, each of which is mapped isometrically into T' . Note that an equivariant morphism does not increase distances.

The notion of a deformation space was introduced by Forester in [4]. By definition, two G -trees are in the same deformation space if they have the same elliptic subgroups, i.e., if a subgroup of G fixes one point in a tree, it also fixes the image of that point in any other tree. Identifying two trees if they differ only by rescaling the metric leads to the projectivized deformation space. Guirardel and Levitt [8] and Clay [2] proved the contractibility of this space. The relative outer space $\text{CV}_n(\mathcal{A})$ is a projectivized deformation space.

Let the *deformation space* \mathcal{D} be the space of simplicial F_n -trees with elliptic subgroups A_1, \dots, A_k . Let $X = (\Gamma, \phi, l) \in \text{CV}_n(\mathcal{A})$. The tree T_1 associated to X is constructed in the following way. Let Γ_0 be the graph obtained by Γ changing the length of the wedge cycles from 0 to a constant $\varepsilon > 0$. Consider the universal cover $\widehat{\Gamma}_0$ of Γ_0 and collapse all the rays that correspond to words $a_{i_1}a_{i_2}a_{i_3}\dots, a_{i_j} \in \mathcal{A}$ and its translates. We will call T_1 an $(\mathcal{A} * B)$ -tree with special vertices or just $(\mathcal{A} * B)$ -tree.

Example 2.10. Consider $\text{Out}(F_2; A)$, where $F_2 = \langle a, b \rangle$, $A = \langle a \rangle$, and consider the point $(\Gamma, \phi = \text{id}, l) \in \text{CV}_2(A)$ consisting of a loop corresponding to b and a vertex corresponding to a . The graph Γ_0 is a rose with two petals. One petal corresponds to a and its length is ε and the other petal correspond to b and its length is 1. In order to construct the tree T_1 associated to this graph, first we consider the universal covering of Γ_0 , and then we collapse the a -axis and its translated axes. Notice that there are infinitely many edges coming out from each vertex (see Figure 1).

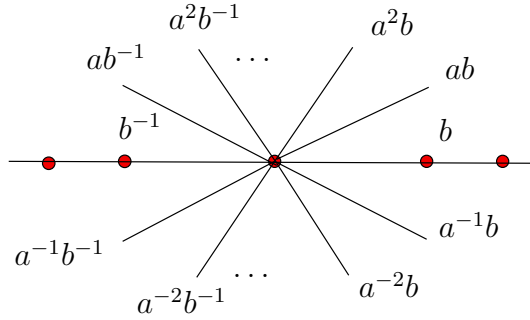


Figure 1: The tree T_1 associated to (Γ, ϕ, l) in Example 2.10.

In [10], we defined the boundary of the relative outer space. We recall the following definition due to Cohen and Lustig.

Definition 2.11. An action of F_n on an \mathbb{R} -tree is *very small* if

1. all edge stabilizers are cyclic (*small action*),
2. $\text{Fix}(g)$ is isometric to a subset of \mathbb{R} for $1 \neq g \in F_n$ (no fixed tripods), and
3. $\text{Fix}(g) = \text{Fix}(g^i)$ for all $i \geq 2$ (no obtrusive powers).

Cohen and Lustig [3] showed that a simplicial action is in $\overline{\text{CV}_n}$ if and only if it is very small. Bestvina and Feighn [1] proved that this is actually true for all actions concluding that the closure of outer space is the set of very small actions of F_n on \mathbb{R} -trees. In the relative case, the main issue is understanding on which \mathbb{R} -trees the group $\mathcal{A} * B$ is acting on and those trees are the $(\mathcal{A} * B)$ -trees with special points. Notice that $\text{CV}_n(\mathcal{A})$ embeds naturally in the space \mathfrak{T} of actions of $\mathcal{A} * B$ on metric \mathbb{R} -trees with special points such that all edge stabilizers are cyclic. As in the case of the standard outer space CV_n , we can endow $\text{CV}_n(\mathcal{A})$ with three topologies: the simplicial (or weak) topology, the length function (or axes) topology, and the Gromov-Hausdorff topology. We already defined the simplicial topology. The length function topology is the coarsest topology making the translation length function $T \mapsto l_T(\phi)$ continuous. The Gromov-Hausdorff topology is defined in the

following way. Given $X \subset T$, $A \subset \text{Out}(F_n; \mathcal{A})$, and $\varepsilon > 0$, a fundamental system of neighborhoods for T , denoted by $U_T(X, A, \varepsilon)$ is the set of trees T' such that there is a map $f : X \rightarrow T'$ satisfying

$$|d(x, \phi \cdot y) - d(f(x), \phi \cdot f(y))| < \varepsilon, \quad \forall x, y \in X, \forall \phi \in A.$$

In the standard CV_n the three topologies are equivalent (see [11]). Because the trees corresponding to the points in $\text{CV}_n(\mathcal{A})$ are *irreducible* (in the sense of [8]), $\text{CV}_n(\mathcal{A})$ with the length function topology is homeomorphic to $\text{CV}_n(\mathcal{A})$ with the Gromov-Hausdorff topology. However, the length function topology of the embedding in the space of actions is not the simplicial topology of $\text{CV}_n(\mathcal{A})$. For example, in the case $n = 2$, $F_2 = \langle a, b \rangle$ and $A = \langle a \rangle$, the modified relative outer space $\text{CV}_2(A)$ is the union of half-open 1-simplices attached to a point X (see Example 2.8). The relative outer space $\text{CV}_2(A)$ endowed with the simplicial topology is not (locally) compact, while $\text{CV}_2(A)$ endowed with the length function topology is a compact space: the two sequences of trees that are the middle-points of the 1-simplices corresponding to the graphs with marking induced by $a \mapsto a$, $b \mapsto a^N b$, as $N \rightarrow \pm\infty$, converge to X . We can consider the closure $\overline{\text{CV}_n(\mathcal{A})}$ of the image of the embedding $\text{CV}_n(\mathcal{A}) \hookrightarrow \mathfrak{T}$. The closure consists of projective classes of actions on metric \mathbb{R} -trees with special points where

1. all edge stabilizers are cyclic,
2. $\text{Fix}(g)$ is isometric to a subset of \mathbb{R} for $1 \neq g \in F_n/\mathcal{A}$,
3. the A_i 's are elliptic elements, and
4. $\text{Fix}(g) = \text{Fix}(g^i)$ for all $i \geq 2$.

The *boundary of the relative outer space* is

$$\partial \text{CV}_n(\mathcal{A}) = \overline{\text{CV}_n(\mathcal{A})} \setminus \text{CV}_n(\mathcal{A}).$$

Our goal is to prove that the dimension of $\partial \text{CV}_n(\mathcal{A})$ is the dimension of $\text{CV}_n(\mathcal{A}) - 1$. From now on, when we talk about the topology of $\text{CV}_n(\mathcal{A})$, $\overline{\text{CV}_n(\mathcal{A})}$, or $\partial \text{CV}_n(\mathcal{A})$ we mean the length function topology. In this topology, $\text{CV}_n(\mathcal{A})$ is contractible and $\overline{\text{CV}_n(\mathcal{A})}$ is contractible (see [8]), and compact (because it is a closed subset of the compact set $\overline{\text{CV}_n(\mathcal{A})}$). However, $\partial \text{CV}_n(\mathcal{A})$ is not compact (see Example 2.8 and the above discussion about the two sequences).

3 (Non-)Geometric Trees with Special Points

Let T be a minimal $(\mathcal{A} * B)$ -tree with special points and length function l . A subtree of T is called *finite* if it is the convex hull of a finite subset. Let $K \subset T$ be a finite subtree such that $K \cap x_i K \neq \emptyset$, for $i \in \{1, \dots, n - \sum s(i)\}$ and $K \cap A_j K \neq \emptyset$, for any $j \in \{1, \dots, k\}$. Let $\mathcal{K} = (K, \{\varphi_i, \varphi^j\})$ be the system with φ_i the restriction of the action of x_i to $x_i^{-1}K \cap K$ and φ^j the restriction of the action of A_j to $A_j^{-1}K \cap K$. By Theorem I.1 [5], there exists a unique $(\mathcal{A} * B)$ -tree such that

1. $T_{\mathcal{K}}$ contains K as an isometrically embedded subtree;

2. If $p \in P_i = x_i^{-1}K \cap K$, then $x_i p = \varphi_i(p)$. If $p \in Q_j = A_j^{-1}K \cap K$ and p is not a special point with elliptic subgroup A_j , then there is $a \in A_j$ such that $p \in a^{-1}K \cap K$, and $ap = \varphi^j(p)$. If p is a special point with elliptic subgroup A_j , $\varphi^j(p) = p$;
3. every orbit of the action meets K ;
4. if T' is another $(\mathcal{A} * B)$ -tree satisfying the first two items, there exists a unique equivariant morphism $f : T_{\mathcal{K}} \rightarrow T'$ such that $f(p) = p$, for $p \in K$.

If the action on T' is (very) small, then the action on $T_{\mathcal{K}}$ is (very) small. Moreover, the tree $T_{\mathcal{K}}$ is not necessarily minimal, but there are arbitrarily large subtrees K such that $T_{\mathcal{K}}$ is minimal (see Section II in [5]).

Definition 3.1. If there exists \mathcal{K} such that $T_{\mathcal{K}} = T$, then T is called *geometric*. Otherwise, T is called *non-geometric*.

Proposition 3.2. Let T be a geometric minimal $(\mathcal{A} * B)$ -tree, i.e., $T = T_{\mathcal{K}}$ for some system \mathcal{K} . We have:

1. Two points $(p_1, \omega_1), (p_2, \omega_2) \in K \times (\mathcal{A} * B)$ define the same point in $T_{\mathcal{K}}$ if and only if

$$p_2 = \varphi_{i_1}^{\varepsilon_1} \cdots \varphi_{i_m}^{\varepsilon_m}(p_1) \text{ such that } \omega_2^{-1}\omega_1 = \alpha_{i_1}^{\varepsilon_1} \cdots \alpha_{i_m}^{\varepsilon_m},$$

where $\varphi_{i_j} \in \{\varphi_i, \varphi^j\}$, $\varepsilon_{i_j} \in \{\pm 1\}$, $\alpha_{i_j} \in \{y_1^1, \dots, y_{s(k)}^k, x_1, \dots, x_{n-\sum_{i=1}^k s(i)}\}$.

2. Let $p_1, p_2 \in K$ and $\omega \in \mathcal{A} * B$. We have $p_2 = \omega p_1$ if and only if

$$p_2 = \varphi_{i_1}^{\varepsilon_1} \cdots \varphi_{i_m}^{\varepsilon_m}(p_1) \text{ with } \omega = \alpha_{i_1}^{\varepsilon_1} \cdots \alpha_{i_m}^{\varepsilon_m},$$

where φ_{i_j} , ε_{i_j} , and α_{i_j} are as above.

Let S be the finite set of vertices of K , $x_i^{-1}K \cap K$, $A_j^{-1}K \cap K$, $\varphi_i(x_i^{-1}K \cap K)$, $\varphi^j(A_j^{-1}K \cap K)$, for all values of i and j . We can list some properties of $T_{\mathcal{K}}$. See [5] for a proof of these facts.

Proposition 3.3. Let $T_{\mathcal{K}}$ as above.

1. If $p \in T_{\mathcal{K}}$ is a branch point, its orbit contains a point of S . The action of $\text{Stab}(p)/\mathcal{A}$ on the set of directions $\pi_0(T_{\mathcal{K}} \setminus \{p\})$ has only finitely many orbits. In particular, there are only finitely many orbits of branch points in $T_{\mathcal{K}}$.
2. If the $(\mathcal{A} * B)$ -action on $T_{\mathcal{K}}$ is small with dense orbits, then every edge stabilizer is trivial.

4 The Index of a Tree with Special Points

We introduce the index of a tree with special points and we find an upper bound for such index. This upper bound will be essential in the computation of the \mathbb{Q} -rank of a tree with special points.

Let T be a minimal $(\mathcal{A} * B)$ -tree with special points. Given $x \in T$, a *direction* d from x is a germ of isometric embeddings $d : [0, \varepsilon] \rightarrow T$ with $d(0) = x$. The subgroup $\text{Stab}(x) < \mathcal{A} * B$ acts on the set

$$D = \{d \mid d \text{ is a direction from } x\}.$$

Note that if $d \in D$, then $\text{Stab}(d)$ is either trivial or infinite cyclic.

Notation 4.1. Let $x \in T$. We denote

$$\text{rk}(\text{ST}(x)) = \begin{cases} \text{rk}(\text{Stab}(x)/\mathcal{A}), & \text{if } x \text{ is not a special point,} \\ \text{rk}(\text{Stab}(x)/\mathcal{A}) + 1, & \text{if } x \text{ is a special point.} \end{cases}$$

Definition 4.2. The *index* of $x \in T$ is

$$i(x) = 2\text{rk}(\text{ST}(x)) + v_1(x) - 2,$$

where $v_1(x)$ is the number of $\text{Stab}(x)$ -orbits of directions from x with trivial stabilizer.

Note that the definition coincides with the definition in [5] if there are no special points, i.e., if $\mathcal{A} = 1$. We will prove that $i(x)$ is finite for any $x \in T$. Notice that if $\text{Stab}(x) = 1$, then $i(x) + 2$ is the cardinality of D . Moreover, we have the following result.

Proposition 4.3. The index $i(x)$ is always non-negative. If $i(x) > 0$, then x is a branch point. Conversely, if the action is very small, then every branch point that is not a special point has $i(x) \geq 1$.

The proof of Proposition 4.3 is similar to the proof of Proposition III.1 [5] considering the three cases: $\text{rk}(\text{ST}(x)) \geq 2$, $\text{rk}(\text{ST}(x)) = 0$, and $\text{rk}(\text{ST}(x)) = 1$.

Remark 4.4. If $\mathcal{A} \neq 1$, for any special point $x \in T$, there is $i \in \{1, \dots, k\}$ such that $A_i < \text{Stab}(x)$, and $\text{rk}(\text{Stab}(x)) \geq \text{rk}(A_i)$.

Let $x \in T$. We denote the $(\mathcal{A} * B)$ -orbit of x by $[x]$. Because $i(x_1) = i(x_2)$ for any $x_1, x_2 \in T$ such that $[x_1] = [x_2]$, the index $i([x])$ is well defined.

Definition 4.5. The *total index* of T is

$$i(T) = \sum_{[x] \in T/(\mathcal{A} * B)} i([x]).$$

Theorem 4.6. Let T be a minimal small $(\mathcal{A} * B)$ -tree.

1. If T is geometric, then

$$i(T) = 2n + 2k - 2 - 2 \sum_{i=1}^k s(i).$$

2. If T is not geometric, then

$$i(T) < 2n + 2k - 2 - 2 \sum_{i=1}^k s(i).$$

Proof. First, assume that T is geometric. Hence, $T = T_{\mathcal{K}}$, for some system \mathcal{K} . Let $[x] \subset T_{\mathcal{K}}$. We define the graph $[x]_{\mathcal{K}}$ with

- vertices $p \in [x] \cap K$;
- an edge x_i from p to $\varphi_i(p)$ if $p \in P_i$ and a loop γ_j if p is elliptic and A_j is the elliptic subgroup acting non-trivially on D .

By Proposition 3.2, $[x]_{\mathcal{K}}$ is connected. We define the weight of an edge e corresponding to x_i and denoted by $w(e)$, to be the valence of its origin $p \in P_i$. The weight of γ_j is 1. All but finitely many edges have weight 2. If Γ is a finite tree and $p \in \Gamma$, we denote by $v_{\Gamma}(p)$ the valence of p in Γ . We define the graph $[x]_{\mathcal{K}}^d$ replacing each vertex p of $[x]_{\mathcal{K}}$ by $v_K(p)$ vertices representing directions from $p \in K$, and replacing each edge by $w(e)$ edges. Let $\pi : [x]_{\mathcal{K}}^d \rightarrow [x]_{\mathcal{K}}$ be the natural projection.

Lemma 4.7. If $p \in [x] \cap K$, then the following statements are true.

1. $\text{rk}(\pi_1([x]_{\mathcal{K}})) = \text{rk}(\text{ST}(p))$.
2. If x is not a special point, $\pi_0([x]_{\mathcal{K}}^d)$ is in one-to-one correspondence with the set of orbits under $\text{Stab}(p)/\mathcal{A}$ of directions d from p in $T_{\mathcal{K}}$. If x is a special point, $\pi_0([x]_{\mathcal{K}}^d)$ is in one-to-one correspondence with γ_j union the set of orbits under $\text{Stab}(p)/\mathcal{A}$ of directions d from p in $T_{\mathcal{K}}$.
3. If x is not a special point and C is a component of $[x]_{\mathcal{K}}^d$, then $\pi_1(C)$ is isomorphic to the corresponding subgroup $\text{Stab}(d) \in \{1, \mathbb{Z}\}$. If x is a special point and C is a component of $[x]_{\mathcal{K}}^d$ not containing γ_j , then $\pi_1(C)$ is isomorphic to the corresponding subgroup $\text{Stab}(d) \in \{1, \mathbb{Z}\}$.

Lemma 4.7 follows from Lemma III.5 [5] and the definition of $[x]_{\mathcal{K}}$. Let G be a finite connected subgraph of $[x]_{\mathcal{K}}$ containing each vertex in S and every edge of weight not equal to 2. Let $G' = \pi^{-1}(G) \subset [x]_{\mathcal{K}}^d$. By Proposition 3.3 and Lemma 4.7, $[x]_{\mathcal{K}}^d$ has a finite number of components. Moreover, we can assume that $\pi_1(G'_j)$ generates $\pi_1(C_j)$, for any component C_j of $[x]_{\mathcal{K}}^d$. As in the proof of Theorem III.2 [5], $\pi_1([x]_{\mathcal{K}})$ is finitely generated and we may assume that $\pi_1(G)$ generates $\pi_1([x]_{\mathcal{K}})$. We can consider $[x] \cap S \neq \emptyset$, otherwise $i([x]) = 0$.

Remark 4.8. If T' is a finite tree and $v_{T'}(p)$ denotes the valence of $p \in T'$, then

$$\sum_{p \in T'} (v_{T'}(p) - 2) = -2.$$

By Lemma 4.7, the definition of G , and $2 - 2\text{rk}(\pi_1(G)) = 2V(G) - 2E(G)$, we have

$$\begin{aligned} i([x]) &= 2\text{rk}(\text{ST}(x)) - 2 + v_1(x) = 2\text{rk}(\pi_1([x]_{\mathcal{K}})) - 2 + \sum_j (1 - \text{rk}(\pi_1([x]_{\mathcal{K}}^d))) \\ &= 2\text{rk}(\pi_1(G)) - 2 + \sum_j (1 - \text{rk}(\pi_1(G'_j))) \\ &= \sum_{p \in V(G)} (v_K(p) - 2) - \sum_{e \in E(G)} (w(e) - 2) \\ &= \sum_{p \in V([x]_{\mathcal{K}})} (v_K(p) - 2) - \sum_{e \in E([x]_{\mathcal{K}})} (w(e) - 2) \\ &= \sum_{p \in [x] \cap K} (v_K(p) - 2) + 2k - \sum_{i=1}^n \sum_{s(i)} \sum_{p \in [x] \cap P_i} (v_{P_i}(p) - 2). \end{aligned}$$

Hence,

$$i(T) = \sum_{[x] \in T/(\mathcal{A} * B)} i([x]) = -2 + 2k + 2n - 2 \sum_{i=1}^k s(i).$$

The case when T is non-geometric can be done, as in the proof of Theorem III.2 [5], approximating T with a sequence of minimal small $(\mathcal{A} * B)$ -trees $T_{\mathcal{K}_m}$ *strongly converging* to the tree T . \square

Corollary 4.9. If T is a minimal very small $(\mathcal{A} * B)$ -tree, the number of orbits of branching points is at most $2n + 2k - 2 - 2 \sum_{i=1}^k s(i)$.

Example 4.10. Consider $\text{Out}(F_2; A)$, where $F_2 = \langle a, b \rangle$, $A = \langle a \rangle$, and the geometric tree T_1 in Figure 1. In this case we have only one orbit of branching points. Let $x \in T_1$ be a branch point. The stabilizer $\text{Stab}(x) = A = \langle a \rangle$. Hence, $\text{rk}(\text{Stab}(x)) = \text{rk}(A) = 1$. Moreover, $v_1(x) = |\{[b], [b^{-1}]\}| = 2$, where $[b^\varepsilon]$ is the A -orbit of the direction associated to b^ε ($\varepsilon \in \{\pm 1\}$). Therefore,

$$i(T_1) = i(x) = 2 \cdot 1 + 2 - 2 = 2 = 2n + 2k - 2 - 2 \sum_{i=1}^k s(i),$$

because $n = 2$, $k = 1$, and $s(1) = 1$.

5 The \mathbb{Q} -rank of a Tree with Special Points

Let $T \neq \mathbb{R}$ be a minimal $(\mathcal{A} * B)$ -tree with non-Abelian length function l . Let $L = L(T) < \mathbb{R}$ be generated by the values of l . The \mathbb{Q} -rank of T is denoted by $r_{\mathbb{Q}}(T)$ and is the dimension of the \mathbb{Q} -vector space $L \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by L . If $r_{\mathbb{Q}}(T)$ is finite, $r_{\mathbb{Q}}(T)$ is the rank r of the Abelian group L and $L/2L \cong (\mathbb{Z}/2\mathbb{Z})^r$. Let $\Lambda = \Lambda(T)$ be the subgroup generated by distances between the branch points.

Proposition 5.1. Let $T \neq \mathbb{R}$ be a minimal $(\mathcal{A} * B)$ -tree with non-Abelian length function and no inversion.

1. Let $x_1, \dots, x_{n - \sum_{i=1}^k s(i)}$ be a basis for B . The numbers

$$l(x_1), \dots, l(x_{n - \sum_{i=1}^k s(i)})$$

generate $L \bmod 2\Lambda$.

2. Let $\{p_j\}_{j \in J}$ be representatives of $(\mathcal{A} * B)$ -orbits of branch points. For $j_0 \in J$, the numbers of $d(p_{j_0}, p_j)$ generate $\Lambda \bmod L$.

Proposition 5.1 is the relative version of Proposition IV.1 [5]. Indeed, in our case $l(y) = 0$, for $y \in A_j$ and $j = 1, \dots, k$.

Corollary 5.2. The group $\Lambda/2\Lambda$ is generated by $n - \sum_{i=1}^k s(i) + b - 1$ elements, where b is the number of orbits of branch points.

Proposition 5.3. 1. Geometric $(\mathcal{A} * B)$ -actions have finite rank.

2. Consider a non-geometric $(\mathcal{A} * B)$ -tree with special points T as the strong limit of a sequence $T_{\mathcal{K}_m}$. If $\liminf_{m \rightarrow +\infty} r(T_{\mathcal{K}_m}) < \infty$, then

$$r_{\mathbb{Q}}(T) \leq \liminf_{m \rightarrow +\infty} r(T_{\mathcal{K}_m})$$

and

$$r_{\mathbb{Q}}(T) < \liminf_{m \rightarrow +\infty} r(T_{\mathcal{K}_m}).$$

Proposition 5.3 is the relative version of Proposition IV.2 [5].

Corollary 5.4. Let T be a geometric minimal $(\mathcal{A} * B)$ -tree without inversions. Let b the number of orbits of branch points. Then $r(T) \leq n - \sum_{i=1}^k s(i) + b - 1$.

Proof. By Proposition 5.3, the action has finite rank r . By Corollary 5.2, $\Lambda/2\Lambda \cong (\mathbb{Z}/2\mathbb{Z})^r$ is generated by $n - \sum_{i=1}^k s(i) + b - 1$ elements. Therefore, $r(T) \leq n - \sum_{i=1}^k s(i) + b - 1$. \square

Theorem 5.5. Let T be a minimal, very small $(\mathcal{A} * B)$ -tree with special points. Then $r_{\mathbb{Q}}(T) \leq 3n + 2k - 3 - 3 \sum_{i=1}^k s(i)$. Equality holds only if the action is simplicial and the only elliptic elements are the A_i 's.

Proof. By Corollary 4.9 and Corollary 5.4, if T is geometric, then $r(T) \leq 3n + 2k - 3 - 3 \sum_{i=1}^k s(i)$. If T is not geometric, there is a sequence of geometric very small trees $T_{\mathcal{K}_m}$ strongly converging to T . By Proposition 5.3, $r_{\mathbb{Q}}(T) < 3n + 2k - 3 - 3 \sum_{i=1}^k s(i)$. We conclude the proof of the theorem proving that if the action is geometric, but there are other elliptic elements beside the A_i 's, then $\Lambda/2\Lambda \cong (\mathbb{Z}/2\mathbb{Z})^r$ with $r < 3n + 2k - 3 - 3 \sum_{i=1}^k s(i)$. There are three different cases.

1. If the action is simplicial, then it is obtained from a graph of groups Γ . Consider the natural (topological) epimorphism $\rho : \mathcal{A} * B \rightarrow \pi_1(\Gamma)$. Since there are other elliptic elements beside the A_i 's, some vertex group is nontrivial and ρ is not injective. Because the free groups are Hopfian, $\text{rk}(\Gamma) < n - \sum_{i=1}^k s(i)$. On the other hand, since there are not inversions, every vertex of Γ is the projection of a branch point of T . By Corollary 4.9, Γ has at most $2n + 2k - 2 - 2 \sum_{i=1}^k s(i)$ vertices. Therefore, Γ has strictly less than $3n + 2k - 3 - 3 \sum_{i=1}^k s(i)$ edges. Because Λ is generated by lengths of edges, $r < 3n + 2k - 3 - 3 \sum_{i=1}^k s(i)$.
2. Suppose that every $(\mathcal{A} * B)$ -orbit is dense in T . In the first case, we had $r < 3n + 2k - 3 - 3 \sum_{i=1}^k s(i)$ because $L/2L$ had 2-rank $< n - \sum_{i=1}^k s(i)$. In this case, we prove that Λ/L has 2-rank $< 2n + 2k - 3 - 2 \sum_{i=1}^k s(i)$, so that $\Lambda/2\Lambda$ has 2-rank $< 3n + 2k - 3 - 3 \sum_{i=1}^k s(i)$. Since T is geometric, $T = T_{\mathcal{K}}$, and we can suppose that every terminal vertex of K is a branch point in T . By Corollary 5.2, if the number of orbits of branch points is $< 2n + 2k - 2 - 2 \sum_{i=1}^k s(i)$, then we are done. Otherwise, let

$$p_1, \dots, p_{2n+2k-2-2 \sum_{i=1}^k s(i)}$$

be representatives of these orbits belonging to K . Each p_j has index 1. By Proposition 3.3, every edge stabilizer is trivial. Therefore, the generators $\{\varphi_i, \varphi^j\}$ are *independent* in the sense of [6]: a reduced word cannot be equal to the identity on a non-degenerate subinterval of K . Denoting by $|\cdot|$ the arclength in K , we have

$$|K| = \sum_{i=1}^{n-\sum s(j)} |P_i| + \sum_{j=1}^k |Q_j|$$

(see [5] and [6]). This gives an equality between numbers of the form $d(q, q')$, where qq' is an edge of K or P_i or Q_j . Since every vertex of K , and so of P_i and Q_j , is a branch point of T , this is an equation in Λ/L . Now, because we have

$$\begin{aligned} d(q, q') &= d(gp_j, hp_m) \\ &= d(gp_j, gp_1) + d(gp_1, gp_m) + d(gp_m, hp_m) \\ &= d(p_j, p_1) + d(p_1, p_m) + l(g^{-1}h) \\ &= d(p_1, p_j) + d(p_1, p_m) \pmod{L}, \end{aligned}$$

we can replace $d(q, q')$ by $d(p_1, p_j) + d(p_1, p_m)$. Hence, we have a relation between the elements of $\{d(p_1, p_j) \mid j = 2, \dots, 2n + 2k - 2 - 2 \sum_{i=1}^k s(i)\}$, whose coefficients are integers mod 2. We need to show that this relation is not trivial. The coefficient of $d(p_1, p_j)$ in the expansion of $|K|$ has the same parity as $\sum v_K(x)$ taken over vertices of K belonging to the orbit of p_j , where $v_K(x)$ is the valence of x in K . The coefficient of $d(p_1, p_j)$ in the expansion of $|P_i|$ (respectively $|Q_j|$) has the same parity as $\sum v_{P_i}(x)$ (respectively $\sum v_{Q_j}(x)$) taken over vertices of P_i (respectively Q_j) belonging to the orbit of p_j , where $v_{P_i}(x)$ is the valence of x in P_i (and $v_{Q_j}(x)$ is the valence of x in Q_j). Since every p_j has index 1, as in the proof of Theorem 4.6, we have the nontrivial relation between the generators of Λ/L :

$$\sum_{j=2}^{2n+2k-2-2 \sum_{i=1}^k s(i)} d(p_1, p_j) = 0 \pmod{L}.$$

3. Assume that the action is not simplicial. Therefore, T may be obtained as a graph of transitive action (see [9]). In particular, there exists a subtree T_v in T such that

- T_v is closed and not equal to a point;
- there is $\delta > 0$ such that, for $g \in \mathcal{A} * B$, $g \in \text{Stab}(T_v)$ or the distance between T_v and gT_v is greater than δ ;
- $\text{Stab}(T_v)$ acts on T_v with dense orbits.

Let T' be the $(\mathcal{A} * B)$ -tree obtained by collapsing each gT_v to a point. The natural action of $\mathcal{A} * B$ on T' is very small. By Theorem 4.6 applied to both T and T' , we have $\text{rk}(\text{Stab}(T_v)) = m < \infty$ and if $i(T_v)$ is the index of T_v with respect to a $\text{Stab}(T_v)$ -tree,

$$0 \leq i(T) - i(T') = i(T_v) - \left(2m + 2k - 2 - 2 \sum_{i=1}^k s(i)\right) \leq 0,$$

where the first inequality is coming from the fact that T is geometric. Hence, $i(T_v) = 2m + 2k - 2 - 2 \sum_{i=1}^k s(i)$ and the action of $\text{Stab}(T_v)$ on T_v is geometric. If there are less than $2m + 2k - 2 - 2 \sum_{i=1}^k s(i)$ distinct $\text{Stab}(T_v)$ -orbits of branch points in T_v , then there are less than $2n + 2k - 2 - 2 \sum_{i=1}^k s(i)$ distinct $(\mathcal{A} * B)$ -orbits in T , and we are done. Otherwise, as in the previous case, we have a nontrivial relation in $\Lambda(T_v)/L(T_v)$, and hence in $\Lambda(T)/L(T)$. □

Recall that our goal is to prove the following theorem.

Main Theorem. The dimension of $\partial \text{CV}_n(\mathcal{A})$ is equal to $\dim(\text{CV}_n(\mathcal{A})) - 1$.

Proof. We know that $\partial \text{CV}_n(\mathcal{A})$ is the set of projective classes of lengths functions of very small actions of F_n such that the A_i 's are elliptic, but those are not the only elliptic elements. By Proposition V.1. [5], if G is a finitely generated group, the space of all projectivized length functions with \mathbb{Q} -rank $\leq N$ has (topological) dimension $\leq N - 1$. By Theorem 5.5, the \mathbb{Q} -rank in the case $G = \text{Out}(F_n; \mathcal{A})$ is $\leq \dim(\text{CV}_n(\mathcal{A}))$. Therefore, $\dim(\partial \text{CV}_n(\mathcal{A})) \leq 3n + 2k - 5 - 3 \sum_{i=1}^k s(i) = \dim(\text{CV}_n(\mathcal{A})) - 1$. Because it is very easy to find a simplex in $\partial \text{CV}_n(\mathcal{A})$ of dimension $3n + 2k - 5 - 3 \sum_{i=1}^k s(i)$, we conclude that $\dim(\partial \text{CV}_n(\mathcal{A})) = \dim(\text{CV}_n(\mathcal{A})) - 1$. □

Example 5.6. In Example 2.8, we described $\text{CV}_2(A)$, where $F_2 = \langle a, b \rangle$, $A = \langle a \rangle$. Obviously, $\dim(\partial\text{CV}_2(A)) = 0 = \dim\text{CV}_2(A) - 1$. Note that in this case the boundary is not connected.

In conclusion, we computed the dimension of $\partial\text{CV}_n(\mathcal{A})$ using the \mathbb{Q} -rank of a tree with special points.

References

- [1] M. Bestvina, M. Feighn, *Outer limits*, preprint (1992), revised March 1994.
- [2] M. Clay, *Contractibility of deformation spaces of G -trees*, Alg. and Geom. Topol. **5** (2005), 1481–1503.
- [3] M. Cohen, M. Lustig, *Very small group actions on \mathbb{R} -trees and Dehn twist automorphisms*, Topology **34** (1995), 575–617.
- [4] M. Forester, *Deformation and rigidity of simplicial group actions on trees*, Geometry and Topology **6** (2002), 219–267.
- [5] D. Gaboriau, G. Levitt, *The rank of actions on \mathbb{R} -trees*, Ann. Sci. de l’École Norm. Sup. (4) **28** (1995), no. 5, 549–570.
- [6] D. Gaboriau, G. Levitt, F. Paulin, *Pseudogroups of isometries of \mathbb{R} and Rip’s theorem on free actions on \mathbb{R} -trees*, Isr. Jour. Math. **87** (1994), 403–428.
- [7] V. Guirardel, *Dynamics of $\text{Out}(F_n)$ on the boundary of Outer space*, Annales Scientifiques de l’École Normale Supérieure (4) **33** (2000), no. 4, 433–465.
- [8] V. Guirardel, G. Levitt, *Deformation spaces of trees*, Groups, Geometry, and Dynamics **1** (2007), 135–181.
- [9] G. Levitt, *Graphs of actions on \mathbb{R} -trees*, Comm. Math. Helv. **69** (1994), 28–38.
- [10] E. Meucci, *Relative Outer Automorphisms of Free Groups*, Ph.D. thesis, University of Utah (2011).
- [11] F. Paulin, *The Gromov topology on \mathbb{R} -trees*, Topology Appl. **32** (1989), 197–221.